Thickness conditions and Littlewood–Paley sets [1]

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Abstract. We consider sets in the real line that have Littlewood–Paley properties LP(p) or LP and study the following question: How thick can these sets be?

References: 14 items.

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Let E be a closed Lebesgue measure zero set in the real line $\mathbb{R}$ and let $I_k,\ k=1,2,\dots,$ be the intervals complimentary to E, i.e., the connected components of the compliment $\mathbb{R}\setminus E$. Call S_k the operator defined by

$$\widehat{(S_k f)} = 1_{I_k}\cdot\widehat{f}, \qquad f\in L^2\cap L^p(\mathbb{R}),$$

where 1_{I_k} is the characteristic function of I_k, and $\widehat{}$ stands for the Fourier transform. Consider the corresponding quadratic Littlewood–Paley function:

$$S(f)=\Big(\sum_k |S_k f|^2\Big)^{1/2}.$$

Following [12] we say that E has property LP(p) ($1<p<\infty$) if for all $f\in L^p(\mathbb{R})$ we have

$$c_1\|f\|_{L^p(\mathbb{R})}\le \|S(f)\|_{L^p(\mathbb{R})}\le c_2\|f\|_{L^p(\mathbb{R})},$$

where c_1, c_2 are positive constants independent of f. In the case when a set has property LP(p) for all p, $1<p<\infty$, we say that it has property LP.

The role of such sets in harmonic analysis and particularly in multiplier theory is well-known. We recall that if G is a locally compact Abelian group and Γ is the group dual to G, then a function $m\in L^\infty(\Gamma)$ is called an L^p -Fourier multiplier, $1\le p\le\infty$, if the operator Q given by

$$\widehat{Qf}=m\cdot\widehat{f}, \qquad f\in L^p\cap L^2(G),$$

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is a bounded operator from $L^p(G)$ to itself (here $\widehat{\ }$ is the Fourier transform on G). The space of all these multipliers is denoted by $M_p(\Gamma)$. Provided with the norm

$$\|m\|_{M_p(\Gamma)} = \|Q\|_{L^p(G)\to L^p(G)},$$

the space $M_p(\Gamma)$ is a Banach algebra (with the usual multiplication of functions). For basic facts on multipliers in the cases when $\Gamma = \mathbb{R},\ \mathbb{Z},\ \mathbb{T}$, where $\mathbb{Z}$ is the group of integers and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the circle, see [1], [13, Chap. IV], [7].

A classical example of an infinite set that has property LP is the set $E = \{\pm 2^k,\ k \in \mathbb{Z}\} \cup \{0\}$ (see, e.g., [13, Chap. IV, Sec. 5]). From arithmetic and combinatorial point of view sets that have property LP(p) or LP were studied extensively, see, e. g., the works [1]–[3], [12]. With the exclusion of [12] these works deal with countable sets, particularly, with subsets of $\mathbb{Z}$. At the same time there exist uncountable sets that have property LP. This fact was first established by Hare and Klemes in [3], see also [8] and [9, Sec. 4].

In this paper we study the following question: How thick can a set $E \subseteq \mathbb{R}$ that has property LP(p) (for some $p,\ p \neq 2$) or property LP be? In Theorems 1, 2 and in Corollary we show that such a set can not be metrically very thick, namely it is porous and the measure of the δ -neighbourhood of any portion of it tends to zero quite rapidly. As a consequence we obtain (see Corollary) an estimate for the Hausdorff dimension of these sets. In Theorem 3 we show that there exist sets which are thin in several senses simultaneously but have LP(p) property for no $p \neq 2$. In Theorem 4 we show that a set can be quite thick but at the same time have property LP. In part our arguments are close to those used by other authors to study subsets of $\mathbb{Z}$ but the mere fact of existence of uncountable (i.e. thick in the sense of cardinality) sets that have property LP brings some specific details to the subject.

It is well-known that a set has property LP(p) if and only if it has property LP(q), where $1/p + 1/q = 1$ (see, e.g., [12]). Thus, it suffices to consider the case when $1 < p < 2$.

We use the following notation. For a set $F \subseteq \mathbb{R}$ we denote its open δ -neighbourhood ($\delta > 0$) by $(F)_\delta$. If F is measurable, then $|F|$ means its Lebesgue measure. A portion of a set $F \subseteq \mathbb{R}$ is a set of the form $F \cap I$, where I is a bounded interval. By $\dim F$ we denote the Hausdorff dimension of F. For basic properties of the Hausdorff dimension we refer the reader to [11]. For a set $F \subseteq \mathbb{R}$ and a point $t \in \mathbb{R}$ we put $F + t = \{x + t : x \in F\}$. By card A we denote the number of elements of a finite set A. By arithmetic progression

of length N we mean a set of the form $\{a+kd,\ k=1,2,\dots,N\}$, where $a,\ d\in\mathbb{R}$ and $d\neq 0$. We use $c,\ c(p),\ c(p,E)$... to denote various positive constants which may depend only on p and the set E.

We recall that a set $F\subseteq\mathbb{R}$ is said to be porous if there exists a constant $c>0$ such that every bounded interval $I\subseteq\mathbb{R}$ contains a subinterval J with $|J|\geq c|I|$ and $J\cap F=\varnothing$.

Theorem 1. *Let $E\subseteq\mathbb{R}$ be a closed set of measure zero. Suppose that E has property* $\mathrm{LP}(p)$ *for some $p,\ p\neq 2$. Then E is porous.*

Earlier Hare and Klemes showed that if a set in $\mathbb{Z}$ has property LP then it is porous [2, Theorem 3.7].

We need certain lemmas.

Lemma 1. *Let $1<p<\infty$. Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be a nonconstant affine mapping. Suppose that a function $m\in M_p(\mathbb{R})$ is continuous at each point of the set $\varphi(\mathbb{Z}^n)$. Then the restriction $m\circ\varphi_{|\mathbb{Z}^n}$ of the superposition $m\circ\varphi$ to $\mathbb{Z}^n$ belongs to $M_p(\mathbb{Z}^n)$, and $\|m\circ\varphi_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)}\leq c\|m\|_{M_p(\mathbb{R})}$, where $c=c(p)>0$ is independent of φ, m and the dimension n.*

Proof. The proof is a trivial combination of the two well-known assertions on multipliers. The first one is the theorem on superpositions with affine mappings [4, Chap. I, Sec. 1.3], which implies that for every function $m\in M_p(\mathbb{R})$ we have $m\circ\varphi\in M_p(\mathbb{R}^n)$ and $\|m\circ\varphi\|_{M_p(\mathbb{R}^n)}=\|m\|_{M_p(\mathbb{R})}$. The second one is the de Leeuw theorem [10] (see also [5]) on restrictions to $\mathbb{Z}^n$, according to which if a function $g\in M_p(\mathbb{R}^n)$ is continuous at the points of $\mathbb{Z}^n$, then its restriction $g_{|\mathbb{Z}^n}$ to $\mathbb{Z}^n$ belongs to $M_p(\mathbb{Z}^n)$ and $\|g_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)}\leq c(p)\|g\|_{M_p(\mathbb{R}^n)}$. The lemma is proved.

Lemma 2. *Let $E\subseteq\mathbb{R}$ be a nowhere dense set and let $F\subseteq\mathbb{R}$ be a finite or countable set. Then for each $\delta>0$ there exists $\xi\in\mathbb{R}$ such that $|\xi|<\delta$ and $(F+\xi)\cap E=\varnothing$.*

Proof. The set
$$\bigcup_{t\in F}(E-t),$$
being a union of at most countable family of nowhere dense sets, can not

contain the whole interval $(-\delta,\ \delta)$, hence there exists $\xi \in (-\delta,\ \delta)$ that does not belong to the union. The lemma is proved.

We say that a (finite or countable) set $F \subseteq \mathbb{R}$ splits a closed set $E \subseteq \mathbb{R}$ if there are no two distinct points of F contained in the same interval complimentary to E.

Lemma 3. *Let $1 < p < 2$. Let $E \subseteq \mathbb{R}$ be a set that has property* $\mathrm{LP}(p)$. *Suppose that F is a subset of an arithmetic progression of length N and splits E. Then* $\operatorname{card} F \le c(p, E) N^{2/q}$, *where* $1/p + 1/q = 1$.

Proof. Consider an arithmetic progression $\{a + kd,\ k = 1, 2, \dots, N\}$. We can assume that $d > 0$. Suppose that a set $F = \{a + k_j d,\ j = 1, 2, \dots, \nu\}$, where $1 \le k_j \le N$, splits E. For $j = 1, 2, \dots, \nu$ let Δ_j be the interval of length δ centered at $a + k_j d$, where $\delta > 0$ is so small that $\delta < d$ and $\Delta_j \cap E = \varnothing,\ j = 1, 2, \dots, \nu$. We put

$$m_\theta = \sum_{j=1}^{\nu} r_j(\theta) \cdot 1_{\Delta_j},$$

where $r_j(\theta) = \operatorname{sign} \sin 2^j \pi\theta,\ \theta \in [0, 1],\ j = 1, 2, \dots,$ are the Rademacher functions.

It is known [12] that if a set E has property $\mathrm{LP}(p)$, then for each function $m \in L^\infty(\mathbb{R})$, whose variations $\operatorname{Var}_{I_k} m$ on the intervals I_k complimentary to E are uniformly bounded, we have $m \in M_p(\mathbb{R})$ and

$$\|m\|_{M_p(\mathbb{R})} \le c(p, E)\Big(\|m\|_{L^\infty(\mathbb{R})} + \sup_k \operatorname{Var}_{I_k} m\Big). \tag{1}$$

Thus we have $\|m_\theta\|_{M_p(\mathbb{R})} \le c$, where $c > 0$ is independent of N and θ. Consider the affine mapping $\varphi(x) = a + dx,\ x \in \mathbb{R}$. Using Lemma 1 in the case when $n = 1$, we see that

$$\|m_\theta \circ \varphi_{|\mathbb{Z}}\|_{M_p(\mathbb{Z})} \le c(p)\|m_\theta\|_{M_p(\mathbb{R})} \le c_1(p).$$

Thus

$$\Big\| \sum_k m_\theta(a + kd) c_k e^{ikx} \Big\|_{L^p(\mathbb{T})} \le c_1(p) \Big\| \sum_k c_k e^{ikx} \Big\|_{L^p(\mathbb{T})}$$

for every trigonometric polynomial $\sum_k c_k e^{ikx}$. In particular,

$$\Big\| \sum_{k=1}^{N} m_\theta(a+kd) e^{ikx} \Big\|_{L^p(\mathbb{T})} \le c_1(p) \Big\| \sum_{k=1}^{N} e^{ikx} \Big\|_{L^p(\mathbb{T})}.$$

Hence,

$$\Big\| \sum_{j=1}^{\nu} r_j(\theta) e^{ik_j x} \Big\|_{L^p(\mathbb{T})} \le c_1(p) \Big\| \sum_{k=1}^{N} e^{ikx} \Big\|_{L^p(\mathbb{T})}. \tag{2}$$

It is easy to verify that

$$\Big\| \sum_{k=1}^{N} e^{ikx} \Big\|_{L^p(\mathbb{T})} \le c(p) N^{1/q},$$

so, (2) yields

$$\int_{\mathbb{T}} \Big| \sum_{j=1}^{\nu} r_j(\theta) e^{ik_j x} \Big|^p \, dx \le c_2(p) N^{p/q}.$$

By integrating this inequality with respect to $\theta \in [0,1]$ and using the Khintchine inequality:

$$\left(\int_0^1 \Big| \sum_j c_j r_j(\theta) \Big|^p d\theta \right)^{1/p} \ge c \left(\sum_j |c_j|^2 \right)^{1/2}, \qquad 1 \le p < 2,$$

(see, e.g., [14, Chap. V, Sec. 8]), we obtain $\nu^{p/2} \le c_3(p) N^{p/q}$. The lemma is proved.

Proof of Theorem 1. We can assume that $1 < p < 2$. For a bounded interval $I \subseteq \mathbb{R}$ let

$$d(I) = \sup\{|J| : J \text{ is an interval, } J \subseteq I, \ J \cap E = \varnothing\}.$$

Suppose that E is not porous. Then, for each positive integer N we can find a (bounded) interval I such that $0 < d(I) < |I|/3N$. Let $d = 2d(I)$. Consider an arithmetic progression $t_k = a + kd, \ k = 1, 2, \dots, N$, that lies in the interior of I. Using Lemma 2, we can find ξ such that $t_k + \xi \notin E, \ k = 1, 2, \dots, N$, and ξ is so small that $\{t_k + \xi, \ k = 1, 2, \dots, N\} \subseteq I$. Note that since $d = 2d(I)$, there no two distinct points of the progression $\{t_k + \xi, \ k = 1, 2, \dots, N\}$ that

lie in the same interval complimentary to E. Thus this progression splits E. By Lemma 3 this is impossible if N is sufficiently large. The theorem is proved.

Theorem 2. *Let $1 < p < 2$. Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that E has property* LP(p)*. Then each portion $E \cap I$ of E satisfies*

$$|(E \cap I)_\delta| \leq c|I|^{2/q}\delta^{1-2/q},$$

where $1/p + 1/q = 1$ and the constant $c = c(p, E) > 0$ is independent of I and δ.

Theorem 2 immediately implies an estimate for the Hausdorff dimension of sets that have LP(p) property. Namely, the following corollary is true.

Corollary. *If $1 < p < 2$ and a set $E \subseteq \mathbb{R}$ has property* LP(p)*, then* $\dim E \leq 2/q$*, where $1/p + 1/q = 1$. Thus, if E has property* LP*, then* $\dim E = 0$.

Proof of Theorem 2. This theorem can be deduced from Theorem 1.3 of the work [12]. We give an independent proof. Consider an arbitrary portion $E \cap I$ of the set E. Let J be the interval concentric with I and of two times larger length. Denote the left-hand endpoint of J by a. Fix a positive integer N and consider the progression $a + kd,\ k = 1, 2, \dots, N$, where $d = |J|/N$. By Lemma 2 one can find ξ such that none of the elements of the progression $\{a + kd + \xi,\ k = 1, 2, \dots, N\}$ is contained in E and $I \subseteq J + \xi = (a + \xi,\ a + Nd + \xi)$.

We define intervals J_k by

$$J_k = (a + (k-1)d + \xi,\ a + kd + \xi), \qquad k = 1, 2, \dots, N.$$

Consider the intervals J_{k_j} such that $J_{k_j} \cap E \neq \varnothing$. Obviously their right-hand endpoints split E, so, by Lemma 3, their number is at most $c(p)N^{2/q}$. Thus the set $E \cap I$ is covered by at most $c(p)N^{2/q}$ intervals of length $d = 2|I|/N$.

Let $\delta > 0$. We can assume that $\delta < |I|$ (otherwise the assertion of the theorem is trivial). Choosing a positive integer N so that

$$\frac{2|I|}{N} \leq \frac{\delta}{3} < \frac{4|I|}{N},$$

we see that the portion $E \cap I$ can be covered by at most $c(p)(12|I|/\delta)^{2/q}$ intervals of length $\delta/3$. It remains to replace each of these intervals with the corresponding concentric interval of nine times larger length. The theorem is proved. The corollary follows.

We note now that a set can be quite thin and at the same time have property $\mathrm{LP}(p)$ for no $p \neq 2$. Consider a set

$$F = \left\{ \sum_{k=1}^{\infty} \varepsilon_k l_k, \ \varepsilon_k = 0 \text{ or } 1 \right\}, \tag{3}$$

where $l_k, \ k = 1, 2, \ldots$, are positive numbers with $l_{k+1} < l_k/2$. It was shown by Sjögren and Sjölin [12] that such sets have property $\mathrm{LP}(p)$ for no $p, \ p \neq 2$. (In particular, the Cantor triadic set does not have property $\mathrm{LP}(p)$ for $p \neq 2$.) Taking a rapidly decreasing sequence $\{l_k\}$ one can obtain a set F of the form (3) such that it is porous and the measure of its δ -neighbourhood rapidly tends to zero. Still, in a sense, any set of the form (3) is thick, it is uncountable and all its points are its accumulation points. Theorem 3 below shows that a set can be thin in several senses simultaneously, and at the same time have property $\mathrm{LP}(p)$ for no $p, \ p \neq 2$.

Theorem 3. *Let ψ be a positive function on an interval $(0, \delta_0), \ \delta_0 > 0$, with $\lim_{\delta \to +0} \psi(\delta)/\delta = +\infty$. There exists a strictly increasing bounded sequence $a_1 < a_2 < \ldots$ such that the set $E = \{a_k\}_{k=1}^{\infty} \cup \{\lim_{k \to \infty} a_k\}$ satisfies the following conditions: 1) E is porous; 2) $|(E)_\delta| \leq \psi(\delta)$ for all sufficiently small $\delta > 0$; 3) E has property $\mathrm{LP}(p)$ for no $p, \ p \neq 2$.*

Proof. Given (real) numbers $a, \ l_1, l_2, \ldots, l_n$ consider the set of all points $a + \sum_{j=1}^{n} \varepsilon_j \, l_j$, where $\varepsilon_j = 0$ or 1. Assume that the cardinality of this set is 2^n. Following [6] we call such a set an n -chain.

We shall need the following refinement of the Sjögren and Sjölin result on the sets (3).

Lemma 4. *Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that for an arbitrary large n the set E contains an n -chain. Then E has property $\mathrm{LP}(p)$ for no $p \neq 2$.*

Proof. Suppose that, contrary to the assertion of the lemma, E has property LP(p) for some $p,\ p \neq 2$. We can assume that $1 < p < 2$.

Let n be such that E contains an n -chain

$$a + \sum_{j=1}^{n} \varepsilon_j\ l_j, \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0; 1\}^n. \tag{4}$$

Consider the set

$$B = \left\{ a + \sum_{j=1}^{n} k_j\ l_j,\ (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \right\}.$$

By Lemma 2 there exists an arbitrary small ξ such that

$$(B + \xi) \cap E = \varnothing \tag{5}$$

Clearly, if ξ is small enough, then no two distinct points of the chain obtained by the same shift ξ of the chain (4) can lie in the same interval complimentary to E. Thus, there exists ξ such that (5) holds and simultaneously the n -chain

$$a + \xi + \sum_{j=1}^{n} \varepsilon_j\ l_j, \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0; 1\}^n,$$

splits E.

For each $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0; 1\}^n$ let I_ε denote the interval complimentary to E that contains the point $a + \xi + \sum_{j=1}^{n} \varepsilon_j\ l_j$. For an arbitrary choice of signs $\pm$ consider the function

$$m = \sum_{\varepsilon \in \{0;1\}^n} \pm 1_{I_\varepsilon}.$$

We have (see (1))

$$\|m\|_{M_p(\mathbb{R})} \leq c, \tag{6}$$

where $c > 0$ is independent of n and the choice of signs.

Consider the following affine mapping φ:

$$\varphi(x) = a + \xi + \sum_{j=1}^{n} x_j\ l_j, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Note that condition (5) implies that the function m is continuous at each point of the set $\varphi(\mathbb{Z}^n)$. Using Lemma 1, we obtain (see (6)) $m\circ\varphi_{|\mathbb{Z}^n} \in M_p(\mathbb{Z}^n)$ and

$$\|m \circ \varphi_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c,$$

where the constant $c > 0$ is independent of n and the choice of signs.

Therefore, for an arbitrary trigonometric polynomial

$$\sum_{k\in\mathbb{Z}^n} c_k e^{i(k,t)}$$

on the torus $\mathbb{T}^n$ we have

$$\Big\| \sum_{k\in\mathbb{Z}^n} m \circ \varphi(k) c_k e^{i(k,t)} \Big\|_{L^p(\mathbb{T}^n)} \leq c \Big\| \sum_{k\in\mathbb{Z}^n} c_k e^{i(k,t)} \Big\|_{L^p(\mathbb{T}^n)}.$$

(We use (k, t) to denote the usual inner product of vectors $k \in \mathbb{Z}^n$ and $t \in \mathbb{T}^n$.) In particular, taking $c_k = 1$ for $k \in \{0; 1\}^n$ and $c_k = 0$ for $k \notin \{0; 1\}^n$, we obtain

$$\Big\| \sum_{\varepsilon\in\{0;1\}^n} m\Big(a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j\Big) e^{i(\varepsilon,t)} \Big\|_{L^p(\mathbb{T}^n)} \leq c \Big\| \sum_{\varepsilon\in\{0;1\}^n} e^{i(\varepsilon,t)} \Big\|_{L^p(\mathbb{T}^n)}.$$

That is

$$\Big\| \sum_{\varepsilon\in\{0;1\}^n} \pm e^{i(\varepsilon,t)} \Big\|_{L^p(\mathbb{T}^n)} \leq c \Big\| \sum_{\varepsilon\in\{0;1\}^n} e^{i(\varepsilon,t)} \Big\|_{L^p(\mathbb{T}^n)}.$$

Raising this inequality to the power p and averaging with respect to the signs $\pm$ (i.e., using the Khintchine inequality), we obtain

$$\Big\| \sum_{\varepsilon\in\{0;1\}^n} e^{i(\varepsilon,t)} \Big\|_{L^2(\mathbb{T}^n)} \leq c \Big\| \sum_{\varepsilon\in\{0;1\}^n} e^{i(\varepsilon,t)} \Big\|_{L^p(\mathbb{T}^n)}. \tag{7}$$

Note that

$$\sum_{\varepsilon\in\{0;1\}^n} e^{i(\varepsilon,t)} = \prod_{j=1}^{n} (1 + e^{it_j}), \quad t = (t_1, t_2, \ldots, t_n) \in \mathbb{T}^n,$$

so (7) yields

$$\|1 + e^{it}\|^n_{L^2(\mathbb{T})} \leq c\|1 + e^{it}\|^n_{L^p(\mathbb{T})}. \tag{8}$$

Since n can be arbitrarily large, relation (8) implies

$$\|1+e^{it}\|_{L^2(\mathbb{T})} \leq \|1+e^{it}\|_{L^p(\mathbb{T})},$$

which, as one can easily verify, is impossible for $1<p<2$. The lemma is proved.

Lemma 5. *Let $l_k,\ k=1,2,\dots,$ be positive numbers satisfying $l_{k+1}<l_k/2$. Then the set F defined by* (3) *contains a strictly increasing sequence $S=\{a_k\}_{k=1}^{\infty}$ such that for every n the sequence S contains an n -chain.*

Proof. For $n=1,2,\dots$ let

$$\alpha_n=\sum_{k=1}^{n^2} l_k, \qquad \beta_n=\sum_{k=1}^{n^2+n} l_k.$$

Clearly $\alpha_1<\beta_1<\alpha_2<\beta_2<\dots$, so the closed intervals $[\alpha_n,\beta_n],\ n=1,2,\dots,$ are pairwise disjoint.

Define sets $F_n\subseteq F,\ n=1,2,\dots,$ as follows

$$F_n=\left\{l_1+l_2+\ldots+l_{n^2}+\sum_{k=n^2+1}^{n^2+n}\varepsilon_k l_k,\ \varepsilon_k=0 \text{ or } 1\right\}.$$

Note that $F_n\subseteq[\alpha_n,\beta_n]$ for all $n=1,2,\dots$.

It remains to put

$$S=\bigcup_{n=1}^{\infty} F_n.$$

The lemma is proved.

We shall now complete the proof of the theorem. Replacing, if needed, the function $\psi(\delta)$ with

$$\widetilde{\psi}(\delta)=\delta\inf_{0<t\leq\delta}\frac{\psi(t)}{t},$$

we can assume that the relation $\psi(\delta)/\delta$ increases to $+\infty$ as δ decreases to zero.

Take a strictly increasing sequence of positive integers $n_k,\ k=1,2,\dots,$ so that

$$6\cdot 2^k\leq\frac{\psi(3^{-n_k})}{3^{-n_k}}, \qquad k=1,2,\dots. \tag{9}$$

Consider the set

$$F = \Big\{ \sum_{k=1}^{\infty} \varepsilon_k 3^{-n_k},\ \varepsilon_k = 0 \text{ or } 1 \Big\}.$$

It is clear that F is porous (as a subset of the Cantor triadic set).

Assuming that $\delta > 0$ is sufficiently small, we can find k such that

$$3^{-n_{k+1}} \leq \delta < 3^{-n_k}. \tag{10}$$

Note that F can be covered by 2^{k+1} closed intervals of length $3^{-n_{k+1}}$. Consider the δ -neighbourhood of each of these intervals. We see that (see (10))

$$|(F)_\delta| \leq 2^{k+1} 3\delta.$$

Hence, taking (9), (10) into account, we obtain

$$|(F)_\delta| \leq \frac{\psi(3^{-n_k})}{3^{-n_k}} \delta \leq \psi(\delta).$$

Using Lemma 5 we can find a strictly increasing sequence $S = \{a_k\}_{k=1}^{\infty}$ contained in F, such that for every n the sequence S contains an n -chain. Let $E = S \cup \{a\}$, where $a = \lim_{k\to\infty} a_k$. It remains to use Lemma 4. The theorem is proved.

Our next goal is to construct a set that has property LP(p) or property LP and at the same time is thick. Theorem 2 implies that if $1 < p < 2$ and a bounded set E has property LP(p), then $|(E)_\delta| = O(\delta^{1-2/q})$ as $\delta \to +0$. Hence, if a bounded set E has property LP, then $|(E)_\delta| = O(\delta^{1-\varepsilon})$ for all $\varepsilon > 0$. The author does not know if these estimates are sharp. A partial solution to this problem is given by Theorem 4 below. This theorem is a simple consequence of the Hare and Klemes theorem [3, Theorem A], which provides a sufficient condition for a set to have property LP(p). Stated for sets in $\mathbb{Z}$ this theorem, as is noted in the end of the work [3], easily transfers to sets in $\mathbb{R}$ and allows to construct perfect sets that have this property. We shall use the version of the Hare and Klemes theorem stated in [9, Sec. 4]. According to this version, for each p, $1 < p < \infty$, there is a constant τ_p $(0 < \tau_p < 1)$ with the following property. Let E be a closed set of measure zero in the interval $[0, 1]$. Suppose that, under an appropriate enumeration, the intervals I_k, $k = 1, 2, \ldots$, complimentary to E in $[0, 1]$ (i.e., the connected components of the compliment $[0, 1] \setminus E$) satisfy

$$\delta_{k+1}/\delta_k \leq \tau_p, \quad k = 1, 2, \ldots, \tag{11}$$

where $\delta_k = |I_k|$. Then E has property LP(p). This in turn implies that if

$$\lim_{k\to\infty} \delta_{k+1}/\delta_k = 0, \tag{12}$$

then E has property LP.

Theorem 4.
(a) *Let* $1 < p < \infty$*. There exists a perfect set* $E \subseteq [0,1]$ *such that it has property* LP(p) *and at the same time* $|(E)_\delta| \geq c\,\delta \log 1/\delta$ *for all sufficiently small* $\delta > 0$*.*

(b) *Let* $\gamma(\delta)$ *be a positive non-decreasing function on* $(0, +\infty)$ *with* $\lim_{\delta\to+0}\gamma(\delta) = 0$*. There exists a perfect set* $E \subseteq [0,1]$ *such that it has property* LP *and at the same time* $|(E)_\delta| \geq c\,\gamma(\delta)\delta \log 1/\delta$*.*

Proof. Let $\delta_k,\ k = 1, 2, \ldots,$ be a sequence of positive numbers with

$$\sum_k \delta_k = 1. \tag{13}$$

Let $E \subseteq [0,1]$ be a closed set. Assume that, under an appropriate enumeration, the intervals $I_k,\ k = 1, 2, \ldots,$ complimentary to E in $[0,1]$ satisfy $|I_k| = \delta_k,\ k = 1, 2, \ldots$. In this case we say that E is generated by the sequence $\{\delta_k\}$. (Certainly $|E| = 0$.) Note that for each sequence $\{\delta_k\}$ of positive numbers with (13) there exists a perfect set $E \subseteq [0,1]$ generated by $\{\delta_k\}$.

It is easy to see that if E is a set generated by a positive sequence $\{\delta_k\}$ satisfying (13), then for all $\delta > 0$ we have

$$|(E)_\delta| \geq 2\delta \operatorname{card}\{k : \delta_k > 2\delta\}. \tag{14}$$

Indeed, if $I_k = (a_k, b_k)$ is an arbitrary interval complimentary to E in $[0,1]$ such that $|I_k| > 2\delta$, then the δ -neighbourhood of E contains the intervals $(a_k, a_k + \delta)$ and $(b_k - \delta, b_k)$.

We shall prove part (a) of the theorem. Fix $p,\ 1 < p < \infty$. Let

$$\delta_k = ae^{-kb}, \quad k = 1, 2, \ldots,$$

where the positive constants a and b are chosen so that conditions (11), (13) hold. Consider a perfect set $E \subseteq [0,1]$ generated by the sequence $\{\delta_k\}$. Using estimate (14), we see that

$$|(E)_\delta| \geq 2\delta\left(\frac{1}{b}\log\frac{a}{2\delta} - 1\right),$$

which proves (a).

Now we shall prove (b). Without loss of generality we can assume that $\gamma(1/e) = 1/4$. Let

$$b(x) = \frac{1}{\gamma(e^{-x})}, \quad x > 0.$$

The function b is non-decreasing, $b(x) \to +\infty$ as $x \to +\infty$, and $b(1) = 4$.

Define a sequence $\{\delta_k\}$ by

$$\delta_k = ae^{-kb(k)}, \quad k = 1, 2, \dots,$$

where the constant $a > 0$ is chosen so that condition (13) holds.

Note that

$$\delta_{k+1}/\delta_k = e^{-((k+1)b(k+1)-kb(k))} \le e^{-b(k)} \to 0, \qquad k \to \infty,$$

and thus, (12) holds.

Consider a perfect set $E \subseteq [0, 1]$ generated by the sequence $\{\delta_k\}$.

Let $\delta > 0$ be sufficiently small. Chose a positive integer $k = k(\delta)$ so that

$$\delta_{k+1} \le 2\delta < \delta_k. \tag{15}$$

We have

$$\operatorname{card}\{k : \delta_k > 2\delta\} \ge k(\delta).$$

So (see (14)),

$$|(E_\delta)| \ge 2\delta k(\delta). \tag{16}$$

Note that (15) implies

$$kb(k) < \log \frac{a}{2\delta} \le (k+1)b(k+1).$$

Hence, for all sufficiently small $\delta > 0$ we have

$$\frac{1}{2}kb(k) < \log \frac{1}{\delta} \le 2(k+1)b(k+1). \tag{17}$$

The left-hand inequality in (17) yields (recall that $b(1) = 4$)

$$2k = \frac{1}{2}kb(1) \le \frac{1}{2}kb(k) < \log \frac{1}{\delta},$$

whence

$$b(2k) \le b\left(\log \frac{1}{\delta}\right) = \frac{1}{\gamma(\delta)}.$$

Combining this inequality and the right-hand inequality in (17), we see that

$$\log \frac{1}{\delta} \le 2(k+1)b(k+1) \le 4kb(2k) \le 4k\frac{1}{\gamma(\delta)}.$$

So,

$$\frac{1}{4}\gamma(\delta)\log \frac{1}{\delta} \le k = k(\delta).$$

Thus (see (16)),

$$|(E)_\delta| \ge \frac{1}{2}\gamma(\delta)\delta \log \frac{1}{\delta}.$$

The theorem is proved.

Remark. As far as the author knows, the question on the existence of a set that has property LP(p) for some $p,\ p \neq 2$, but does not have property LP is open.

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